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The initial value problem, scattering and inverse scattering, for Schrödinger equations with a potential and a non-local nonlinearity*

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Abstract

We consider nonlinear Schrödinger equations with a potential, and non-local nonlinearities, that are models in mesoscopic physics, for example of a quantum capacitor, and that are also models of molecular structure. We study in detail the initial value problem for these equations, in particular, existence and uniqueness of local and global solutions, continuous dependence on the initial data and regularity. We allow for a large class of unbounded potentials. We have no restriction on the growth at infinity of the positive part of the potential. We also construct the scattering operator in the case of potentials that go to zero at infinity. Furthermore, we give a method for the unique reconstruction of the potential from the small amplitude limit of the scattering operator. In the case of the quantum capacitor, our method allows us to uniquely reconstruct all the physical parameters from the small amplitude limit of the scattering operator.

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1. Introduction

In recent years there has been considerable interest in nonlinear Schrödinger equations with a potential (NLSP) and a non-local nonlinearity that is concentrated in a bounded region of space, for example, to model physical situations that appear in mesoscopic physics. In particular, in [18–20, 27, 28] the following equation was introduced to model a quantum

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capacitor (in [19] more general equations are discussed).

$$\mathrm{i}\frac{\partial}{\partial t}u(x,t) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x,t) + V_0(x)u(x,t) + \lambda Q(u)\chi_{[b,c]}(x)u(x,t), \qquad u(x,0) = \varphi(x), \tag{1.1}$$

with, $x, t \in \mathbb{R}$, and where we have set Planck's constant $\hbar$ equal to 1 and the mass equal to 1/2. For any $O \in \mathbb{R}$, χ_O is the characteristic function of O. The external potential, V_0, is a double barrier,

$$V_0(x) = \beta[\chi_{[a,b]}(x) + \chi_{[c,d]}(x)], \qquad \beta > 0,$$

where, $a < b < c < d$. Furthermore,

$$Q(u) := \int_b^c |u(x,t)|^2\,\mathrm{d}x$$

is the dimensionless electric charge trapped in the well $[b, c]$.

We briefly describe the physical aspects of the model given by (1.1), following [20]. As is well known [7], the interaction between electrons can play a crucial role in the electrical transport properties of mesoscopic systems. In the case considered in [20, 27] a cloud of electrons move in a double barrier heterostructure in which the well region confined between the two potential barriers acts like a quantum capacitor whose energy depends on the electron charge trapped inside it. The localization of the interaction is justified by the existence of a resonant state that allows for a long sojourn time of the electrons inside the well. This leads to an accumulation of electric charge inside the quantum capacitor. A main feature of equation (1.1) is that the nonlinearity is concentrated only in the region where the resonant state is localized, i.e., within the two barriers. In [20], among other results, a one mode approximation was considered, an adiabatic condition was introduced, and numerical simulations where performed.

In this paper we study the following generalization of (1.1),

$$\mathrm{i}\frac{\partial}{\partial t}u(x,t) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x,t) + V_0(x)u(x,t) + \lambda(V_1u,u)V_2(x)u(x,t), \qquad u(x,0) = \varphi(x), \tag{1.2}$$

$x, t \in \mathbb{R}$. By $(\cdot,\cdot)$ we denote the L^2 scalar product. The external potential V_0 is real valued, and $V_1(x)$, $V_2(x)$ are, in general, complex valued. $\lambda \in \mathbb{C}$ is a coupling constant.

In the particular case where V_0 is a double well, $V_1 = V_2$ is real valued, and $\lambda \in \mathbb{R}$, this equation was extensively studied as a model for molecular structure [10, 11, 16, 17, 32–34] and the references quoted there. In these papers, molecular localization, the suppression of the beating effect by the nonlinearity and the semi-classical limit were studied, among other problems.

Below, we study in a detailed way the initial value problem for (1.2). We prove existence and uniqueness of local solutions in L^2, $\mathbb{H}^1_Q$, and $\mathbb{H}^2_Q$, where $\mathbb{H}^j_Q$, $j = 1, 2$, are Sobolev spaces, and $\mathbb{H}^2_Q \subset \mathbb{H}^1_Q \subset L^2$ (see section 2 for the definition of $\mathbb{H}^j_Q$, $j = 1, 2$). We also prove the continuity of the solution on the initial value and a regularity result that tells us that if the initial value belongs to $\mathbb{H}^1_Q$, the solution in $\mathbb{H}^1_Q$ cannot blow-up before the solution in L^2 does. We obtain also a similar result on regularity between $\mathbb{H}^1_Q$ and $\mathbb{H}^2_Q$ solutions, when the initial data belong to $\mathbb{H}^2_Q$. Furthermore, we prove that if V_1, V_2 are real valued and $\lambda \in \mathbb{R}$ the L^2 solutions are global, i.e., they exist for all times $t \in \mathbb{R}$. If moreover, $V_1 = V_2$ we prove that the $\mathbb{H}^j_Q$, $j = 1, 2$, solutions are global. We prove these results for a large class of unbounded external potentials, V_0 (see section 2). In fact, we have no restriction on the growth at infinity on the positive part of the potential V_0. These results prove that (1.2) forms a dynamical system by generating a continuous local/global flow [22]. In this sense, the spaces $L^2, \mathbb{H}^1_Q$ and $\mathbb{H}^2_Q$ are fundamental for equation (1.2).

In [17, 34] the existence and uniqueness of global solutions in the Sobolev spaces $\mathbb{H}^j$, $j = 1, 2$, was proven in the case where the external potential V_0 is bounded.

Then, we consider external potentials that decay at infinity, and we construct the scattering operator for (1.2) with reference dynamics given by the self-adjoint realization in L^2 of $-\frac{d^2}{dx^2}$. Furthermore, we study the inverse scattering problem. We prove that the small amplitude limit of the scattering operator allows us to uniquely reconstruct V_0 and λ. In the particular case of the quantum capacitor (1.1), this gives us all the physical parameters.

There is a very extensive literature on the initial value problem and in scattering for the nonlinear Schrödinger equation without a potential and local nonlinearities. As general references see, for example [8, 15, 22, 29, 38] and [39]. For the case of nonlinearities concentrated in a finite number of points see [1–4].

For the initial value problem for the NLSP with a local nonlinearity and a potential that has bounded second derivative see [22]. For the forced problem on the half-line see [49], where for local solutions there is no restriction on the growth of the positive part of the potential at infinity, and for global solutions only mild restrictions that allow, for example, for exponential growth.

For direct and inverse scattering for the NLSP with local nonlinearities see [41, 44] and [45], and for the case of the nonlinear Klein–Gordon equation with a potential and local nonlinearities see [42] and [46]. For an expository review of these results see [47]. For the forced NLSP on the half-line see [48].

For centre manifolds for the NLSP with local nonlinearities see [43], and for the nonlinear Schrödinger equation with a double-well potential and a cubic local nonlinearity see [35].

The paper is organized as follows. In section 2 we prove our results on the initial value problem. In section 3 we construct the scattering operator (direct scattering) and we prove our results on inverse scattering.

2. The initial value problem

In this section we absorb the coupling constant λ into V_2 and we consider the initial value problem for the following NLSP,

$$i\frac{\partial}{\partial t}u(x,t) = -\frac{d^2}{dx^2}u(x,t) + V_0(x)u(x,t) + (V_1u,u)V_2u(x,t), \qquad u(x,0) = \varphi(x). \tag{2.1}$$

By $W_{j,p}$, $j = 1, 2, \ldots, 1 \leqslant p \leqslant \infty$, we denote the Sobolev space [5] of all functions in L^p such that all its derivatives of order up to j are functions in L^p. By $\|\cdot\|_{j,p}$ we denote the norm in $W_{j,p}$. By $\mathbb{H}^j$, $j = 1, 2, \ldots$, we denote the L^2 based Sobolev spaces, $\mathbb{H}^j := W_{j,2}$.

We suppose that the potential V_0 satisfies the following assumption.

Assumption A. *Assume that*

$$V_0 = V_{0,1} + V_{0,2}, \qquad \text{with} \qquad V_{0,j} \in L^1_{\text{loc}}, \quad V_{0,1} \geqslant 0, \tag{2.2}$$

$$\sup_{x\in\mathbb{R}} \int_x^{x+1} |V_{0,2}(x)|\, dx < \infty. \tag{2.3}$$

We denote $Q := \sqrt{V_{0,1}}$ and by $D(Q)$ the domain in L^2 of the operator of multiplication by Q.

Let us denote by H_0 the self-adjoint realization of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ in L^2 with domain $\mathbb{H}^2$. Equation (2.3) implies that $V_{0,2}$ is a quadratic form bounded perturbation of H_0 with relative bound zero (this is proven, for example, as in the proof of [49, proposition 2.1]), that is, for any $\epsilon > 0$ there is a constant K_ϵ such that

$$|(V_{0,2}\varphi, \varphi)| \leqslant \epsilon \|\varphi'\|_{L^2}^2 + K_\epsilon \|\varphi\|_{L^2}^2. \tag{2.4}$$

It follows that the quadratic form

$$h(\varphi, \psi) := (\varphi', \psi') + (V_0\varphi, \psi), \qquad \text{with domain,} \quad D(h) := \mathbb{H}^1 \cap D(Q), \tag{2.5}$$

is closed and bounded from below. Let H be the associated bounded-below, self-adjoint operator (see, [21, 31]). Then,

$$D(H) = \left\{\varphi \in D(h) : -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi + V\varphi \in L^2\right\} \quad \text{and} \quad H\varphi = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi + V\varphi, \quad \varphi \in D(H). \tag{2.6}$$

Let us take $N > 0$ such that $H + N > 1$ and let us denote $D := \cap_{m=-\infty}^{\infty} D((H+N)^m)$. For any $s \in \mathbb{R}$ let $\mathbb{H}_Q^s$ be the completion of D in the norm, $\|(H+N)^{s/2}\|_{L^2}$. Note that $\mathbb{H}_Q^0 = L^2, \mathbb{H}_Q^1 = D(\sqrt{H+N}) = D(h)$ and that $\mathbb{H}_Q^2 = D(H)$.

Observe that the following norm is equivalent to the norm of $\mathbb{H}_Q^1$:

$$\max[\|\varphi\|_{\mathbb{H}^1}, \|Q\varphi\|_{L^2}].$$

Furthermore, if $V_{0,1} = 0$ the potential V_0 is a quadratic form bounded perturbation of H_0 with relative bound zero, and then, $\mathbb{H}_Q^1 = \mathbb{H}^1$. In this case the $\mathbb{H}_Q^1$ solutions that we consider below are just $\mathbb{H}^1$ solutions.

As $\mathbb{H}_Q^2 = D(H)$, the following norm is equivalent to the norm of $\mathbb{H}_Q^2$:

$$\max\left[\|\varphi\|_{\mathbb{H}_Q^1}, \left\|\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0\right)\varphi\right\|_{L^2}\right].$$

The paper [23] gives sufficient conditions in V_0 that assure that $\mathbb{H}_Q^2 \subset \mathbb{H}^2$. Furthermore, if

$$\sup_{x\in\mathbb{R}} \int_x^{x+1} |V_0(x)|^2 \,\mathrm{d}x < \infty, \tag{2.7}$$

it follows from (2.4), and as $\|V_0\varphi\|_{L^2}^2 = \left(V_0^2\varphi, \varphi\right)$, and $(\varphi', \varphi') \leqslant \|H_0\varphi'\|^2 + \|\varphi\|^2$, that V_0 is relatively bounded with respect to H_0 with relative bound zero, that is, for any $\epsilon > 0$ there is a constant K_ϵ such that

$$\|V_0\varphi\|_{L^2} \leqslant \epsilon \|H_0\varphi\|_{L^2} + K_\epsilon \|\varphi\|_{L^2}. \tag{2.8}$$

In this case (see [21, 31]) $\mathbb{H}_Q^2 = \mathbb{H}^2$ and the $\mathbb{H}_Q^2$ solutions that we consider below are just $\mathbb{H}^2$ solutions.

It follows from the functional calculus of self-adjoint operators that H is bounded from $\mathbb{H}_Q^s$ to $\mathbb{H}_Q^{s-2}$ and that $\mathrm{e}^{-\mathrm{i}tH}$ is a strongly-continuous unitary group on $\mathbb{H}_Q^s$, $s \in \mathbb{R}$. Moreover, for any $\varphi \in \mathbb{H}_Q^s$, $s \in \mathbb{R}$, $\mathrm{e}^{-\mathrm{i}tH}\varphi \in C\left(\mathbb{R}, \mathbb{H}_Q^s\right) \cap C^1\left(\mathbb{R}, \mathbb{H}_Q^{s-2}\right)$ and

$$\mathrm{i}\frac{\partial}{\partial t}\mathrm{e}^{-\mathrm{i}tH}\varphi = H\,\mathrm{e}^{-\mathrm{i}tH}\varphi = \mathrm{e}^{-\mathrm{i}tH}H\varphi, \qquad \varphi \in \mathbb{H}_Q^s.$$

We introduce some further notation that we use below. Let $I =: [0, T]$ if $0 < T < \infty$ and $I = [0, \infty)$ if $T = \infty$. For any Banach space $\mathcal{X}$ we denote by $\mathcal{X}_R$ the closed ball in $\mathcal{X}$ with centre zero and radius R. If $T < \infty$ we denote by $C(I, \mathcal{X})$ the Banach space of continuous functions from I into $\mathcal{X}$ and if $T = \infty$ we denote by $C_B(I, \mathcal{X})$ the Banach space of continuous and bounded functions from I into $\mathcal{X}$. For $T < \infty$, we define $\mathcal{N} := C(I, L^2)$ and $\mathcal{N}^j := C\big(I, \mathbb{H}^j_Q\big)$, $j = 1, 2$. For functions $u(t, x)$ defined in $\mathbb{R}^2$ we denote $u(t)$ for $u(t, \cdot)$.

We study the initial value problem (2.1) for $t \geqslant 0$, but by changing t into $-t$ and taking the complex conjugate of the solution (time reversal) we also obtain the results for $t \leqslant 0$.

By a L^2 solution on I to (2.1) we mean a function $u \in C(I, L^2) \cap C^1\big(I, \mathbb{H}^{-2}_Q\big)$ that satisfies (2.1).

Multiplying both sides of (2.1) (evaluated at τ) by $\mathrm{e}^{-\mathrm{i}(t-\tau)H}$ and integrating in τ from zero to t we obtain that

$$u(t) = \mathrm{e}^{-\mathrm{i}tH}\varphi + \frac{1}{\mathrm{i}}GF(u), \qquad \text{with} \qquad F(u) := (V_1u, u)V_2u \tag{2.9}$$

and where

$$Gu := \int_0^t \mathrm{e}^{-\mathrm{i}(t-\tau)H}u(\tau)\,\mathrm{d}\tau. \tag{2.10}$$

Moreover, let $u \in C(I, L^2)$ be a solution to (2.9). Then, it follows from (2.9) that $u \in C^1\big(I, \mathbb{H}^{-2}_Q\big)$. We prove that u solves (2.1) taking the derivative of both sides of (2.9). Hence, equations (2.1) and (2.9) are equivalent. We obtain our results below solving the integral equation (2.9).

In the next theorem we prove the existence of local solutions in L^2.

Theorem 2.1. *Suppose that assumption A is satisfied and that $V_j \in L^\infty$, $j = 1, 2$. Then, for any $\varphi \in L^2$ there is $< T < \infty$ such that (2.1) has a unique solution $u \in C(I, L^2)$ with $u(0) = \varphi$. T depends only on $\|\varphi\|_{L^2}$.*

Proof. We define

$$\mathcal{C}(u) : \mathrm{e}^{-\mathrm{i}tH}\varphi + \frac{1}{\mathrm{i}}GF(u). \tag{2.11}$$

We will prove that we can take R large enough, and T so small that $\mathcal{C}$ is a contraction on $\mathcal{N}_R$.

It follows from the Schwarz inequality that for any $u, v \in \mathcal{N}_R$,

$$\|F(u) - F(v)\|_{\mathcal{N}} \leqslant C\big(\|u\|^2_{\mathcal{N}} + \|v\|^2_{\mathcal{N}}\big)\|u - v\|_{\mathcal{N}} \leqslant C2R^2\|u - v\|_{\mathcal{N}}. \tag{2.12}$$

Then, as $\mathrm{e}^{-\mathrm{i}tH}$ is unitary on L^2,

$$\|\mathcal{C}(u)\|_{\mathcal{N}} \leqslant \big[\|\varphi\|_{L^2} + CT\|u\|^3_{\mathcal{N}}\big] \leqslant [\|\varphi\|_{L^2} + CTR^3], \tag{2.13}$$

$$\|\mathcal{C}(u) - \mathcal{C}(v)\|_{\mathcal{N}} \leqslant CT\big(\|u\|^2_{\mathcal{N}} + \|v\|^2_{\mathcal{N}}\big)\|u - v\|_{\mathcal{N}} \leqslant CT2R^2\|u - v\|_{\mathcal{N}}. \tag{2.14}$$

Then, we can take R, T such that $\|\varphi\|_{L^2} + CTR^3 \leqslant R$ and $d := CT2R^2 < 1$, what makes $\mathcal{C}$ a contraction on $\mathcal{N}_R$. By the contraction mapping theorem [30] $\mathcal{C}$ as a unique fixed point, u, in $\mathcal{N}$ that is a solution to (2.9).

Suppose that there is another solution $v \in \mathcal{N}$. By the argument above, we have that $v(t) = u(t)$ for $t \in [0, T_0]$ for some $T_0 \leqslant T$. By iterating this argument we prove that $v(t) = u(t), 0 \leqslant t \leqslant T$. □

We now prove that the solution depends continuously on the initial data.

Theorem 2.2. *Suppose that assumption A is satisfied and that $V_j \in L^\infty$, $j = 1, 2$. Then, the solution $u \in C([0, T], L^2), 0 < T < \infty$, to (2.1) with $u(0) = \varphi$, given by theorem 2.1, depends continuously on the initial value φ. In a precise way, let $\varphi_n \to \varphi$ strongly in L^2. Then, for n large enough, the solutions $u_n \in C([0, T], L^2)$ to (2.1) with initial values φ_n exist and $u_n \to u$ in $C([0, T], L^2)$.*

Proof. We first prove a local version of the theorem with T replaced by a T_0 small enough. We define

$$\mathcal{C}_n(u) := \mathrm{e}^{-\mathrm{i}tH}\varphi_n + \frac{1}{\mathrm{i}} GF(u), \quad u \in \mathcal{N}(T_0) := C([0, T_0], L^2). \tag{2.15}$$

As $\mathrm{e}^{-\mathrm{i}tH}\varphi_n \to \mathrm{e}^{-\mathrm{i}tH}\varphi$ in $\mathcal{N}(T_0)$, for n large enough $\mathcal{C}_n$ and $\mathcal{C}$ are contractions in $\mathcal{N}_R(T_0)$ with the same R, T_0, d. The unique fixed points, u_n, u, are solutions to (2.1) in $C([0, T_0], L^2)$ with, respectively, $u_n(0) = \varphi_n, u(0) = \varphi$. Moreover, as $\mathcal{C}_n(u_n) - \mathcal{C}(u) = \mathcal{C}_n(u_n) - \mathcal{C}_n(u) + \mathcal{C}_n(u) - \mathcal{C}(u)$,

$$\|u_n - u\|_{\mathcal{N}(T_0)} = \|\mathcal{C}_n(u_n) - \mathcal{C}(u)\|_{\mathcal{N}(T_0)} \leqslant \mathrm{d}\|u_n - u\|_{\mathcal{N}(T_0)} + \|\varphi_n - \varphi\|_{L^2}, \tag{2.16}$$

and as $d < 1$, $u_n \to u$ in $\mathcal{N}(T_0)$. As the interval of existence of the solution given in theorem 2.1 depends only on the L^2 norm of the initial value, we can extend this argument, step by step, to the whole interval $[0, T]$. □

Remark 2.3. Let T_m be the maximal time such that the solution u given in theorem 2.1 can be extended to a solution $u \in C([0, T_m), L^2)$ with $u(0) = \varphi$. Then, if T_m is finite we must have that $\lim_{t\uparrow T_m} \|u(t)\|_{L^2} = \infty$. In other words, the solution exists for all times unless it blows up in the L^2 norm for some finite time. To prove this result suppose that $\|u(t)\|_{L^2}$ remains bounded as $t \uparrow T_m$. Then, by theorem 2.1 we can extend the solution continuously to $T_m + \epsilon$ for some $\epsilon > 0$, contradicting the definition of T_m.

Another consequence of theorem 2.1 is that (2.1) has at most one solution in $C(I, L^2)$ with $u(0) = \varphi$. Suppose, on the contrary, that there are two u_1, u_2. Then, by theorem 2.1, $u_1(t) = u_2(t), t \in [0, T_0]$, for some $0 < T_0 < T$. Let T_m be the maximal time such that $u_1(t) = u_2(t), t \in [0, T_m)$. Consider first the case where $T < \infty$. Then, $T_m = T$, because if $T_m < T$, theorem 2.1 would imply that $u_1(t) = u_2(t)$ for $t \leqslant T_m + \epsilon$ for some $\epsilon > 0$, in contradiction with the definition of T_m. Then, $T_m = T$ and by continuity, $u_1(T) = u_2(T)$, completing the proof in the case $T < \infty$. A similar argument proves that if $T = \infty$, T_m cannot be finite.

We will use the uniqueness of L^2 solutions given by theorem 2.1 and remark 2.3 in the construction of the scattering operator in theorem 3.1 in section 3.

We now study solutions in $\mathbb{H}^1_Q$.

Theorem 2.4. *Suppose that assumption A holds that V_1 satisfies (2.3) and that $V_2 \in W_{1,\infty}$. Then, for any $\varphi \in \mathbb{H}^1_Q$ there is a $0 < T < \infty$ such that (2.1) has a unique solution $u \in C\big([0, T], \mathbb{H}^1_Q\big)$, with $u(0) = \varphi$. T depends only on $\|\varphi\|_{\mathbb{H}^1_Q}$.*

Proof. As V_1 satisfies (2.3), it follows from (2.4) that

$$|(V_1\varphi, \varphi)| \leqslant C\|\varphi\|^2_{\mathbb{H}^1}, \quad \text{for all } \varphi \in \mathbb{H}^1. \tag{2.17}$$

Then, defining $\mathcal{C}$ as in (2.11) and as $V_2 \in W_{1,\infty}$,

$$\|\mathcal{C}(u)\|_{\mathcal{N}^1} \leqslant \|\varphi\|_{\mathbb{H}^1_Q} + CT\|u\|^3_{\mathcal{N}^1}, \tag{2.18}$$

$$\|\mathcal{C}(u) - \mathcal{C}(v)\|_{\mathcal{N}^1} \leqslant CT\left(\|u\|^2_{\mathcal{N}^1} + \|v\|^2_{\mathcal{N}^1}\right)\|u - v\|_{\mathcal{N}^1}. \tag{2.19}$$

We take R, T such that $\|\varphi\|_{\mathbb{H}^1_Q} + CTR^3 \leqslant R$ and, $d := CT2R^2 < 1$. By (2.18) and (2.19), with this choice $\mathcal{C}$ is a contraction on $\mathcal{N}^1_R$ with the contraction rate d. The unique fixed point, u, is a solution to (2.1) with $u(0) = \varphi$. We prove the uniqueness of the solution in $C\big([0, T], \mathbb{H}^1_Q\big)$ as in the proof of theorem 2.1. □

There is also continuous dependence of the solutions in $\mathbb{H}^1_Q$.

Theorem 2.5. *Suppose that assumption A holds that V_1 satisfies (2.3) and that $V_2 \in W_{1,\infty}$. Then, the solution $u \in C\big([0,T], \mathbb{H}^1_Q\big)$, $0 < T < \infty$, to (2.1) with $u(0) = \varphi$, given by theorem 2.4, depends continuously on the initial value φ. In a precise way, let $\varphi_n \to \varphi$, strongly in $\mathbb{H}^1_Q$. Then, for n large enough, the solutions $u_n \in C\big([0,T], \mathbb{H}^1_Q\big)$ to (2.1) with initial values φ_n exist and $u_n \to u$ in $C\big([0,T], \mathbb{H}^1_Q\big)$.*

Proof. The theorem is proven as in the proof of theorem 2.2 replacing in the argument $\mathcal{N}(T_0)$ by $C\big([0,T_0], \mathbb{H}^1_Q\big)$. □

Remark 2.6. We prove as in remark 2.3 that the solution to (2.1) in $\mathbb{H}^1_Q$ exists for all $t > 0$ unless it blows up in the $\mathbb{H}^1_Q$ norm for some finite time and that theorem 2.4 implies that (2.1) has at most one solution on $C\big(I, \mathbb{H}^1_Q\big)$.

Let us now study solutions in $\mathbb{H}^2_Q$.

Theorem 2.7. *Suppose that assumption A holds that V_1 satisfies (2.3) and that $V_2 \in W_{2,\infty}$. Then, for any $\varphi \in \mathbb{H}^2_Q$ there is a $0 < T < \infty$ such that (2.1) has a unique solution $u \in C\big([0,T], \mathbb{H}^2_Q\big)$, with $u(0) = \varphi$. T depends only on $\|\varphi\|_{\mathbb{H}^2_Q}$.*

Proof. As $V_2 \in W_{2,\infty}$, for some constant C,

$$\|V_2\varphi\|_{\mathbb{H}^2_Q} \leqslant C\|\varphi\|_{\mathbb{H}^2_Q}, \quad \text{for all} \quad \varphi \in \mathbb{H}^2_Q. \tag{2.20}$$

Then, with $\mathcal{C}$ defined as in (2.11),

$$\|\mathcal{C}(u)\|_{\mathcal{N}^2} \leqslant \|\varphi\|_{\mathbb{H}^2_Q} + CT\|u\|^3_{\mathcal{N}^2}, \tag{2.21}$$

$$\|\mathcal{C}(u) - \mathcal{C}(v)\|_{\mathcal{N}^2} \leqslant CT\big(\|u\|^2_{\mathcal{N}^2} + \|v\|^2_{\mathcal{N}^2}\big)\|u - v\|_{\mathcal{N}^2}. \tag{2.22}$$

Let R, T be such that $\|\varphi\|_{\mathbb{H}^2_Q} + CTR^3 \leqslant R$ and $d := CT2R^2 < 1$. Hence, it follows from (2.21) and (2.22) that $\mathcal{C}$ is a contraction on $\mathcal{N}^2_R$ with the contraction rate d. The unique fixed point, u, is a solution to (2.1) with $u(0) = \varphi$. We prove the uniqueness of the solution in $C\big([0,T], \mathbb{H}^2_Q\big)$ as in the proof of theorem 2.1. □

Theorem 2.8. *Suppose that assumption A holds that V_1 satisfies (2.3) and that $V_2 \in W_{2,\infty}$. Then, the solution $u \in C\big([0,T], \mathbb{H}^2_Q\big)$, $0 < T < \infty$, to (2.1) with $u(0) = \varphi$, given by theorem 2.7, depends continuously on the initial value φ. In a precise way, let $\varphi_n \to \varphi$, strongly in $\mathbb{H}^2_Q$. Then, for n large enough, the solutions $u_n \in C\big([0,T], \mathbb{H}^2_Q\big)$ to (2.1) with initial values φ_n exist and $u_n \to u$ in $C\big([0,T], \mathbb{H}^2_Q\big)$.*

Proof. The theorem is proven as in the proof of theorem 2.2 replacing in the argument $\mathcal{N}(T_0)$ by $C\big([0,T_0], \mathbb{H}^2_Q\big)$. □

Remark 2.9. We prove as in remark 2.3 that the solution to (2.1) in $\mathbb{H}^2_Q$ exists for all $t > 0$ unless it blows up in the $\mathbb{H}^2_Q$ norm for some finite time and that theorem 2.7 implies that (2.1) has at most one solution on $C\big(I, \mathbb{H}^2_Q\big)$.

We now consider the problem of the regularity of solutions. Suppose that the conditions of theorems 2.1 and 2.4 are satisfied and that $\varphi \in \mathbb{H}^1_Q$. Then, by theorem 2.1, (2.1) has a unique L^2 solution and by theorem 2.4 a unique $\mathbb{H}^1_Q$ solution, both with initial value φ. In the proposition below we prove that it is impossible that the $\mathbb{H}^1_Q$ solution blows-up before the L^2 solution.

Proposition 2.10. *Suppose that assumption A holds that $V_1 \in L^\infty$ and that $V_2 \in W_{1,\infty}$. Let $u \in C([0,T], L^2)$, $0 < T < \infty$, be a solution to (2.1) with $u(0) = \varphi \in \mathbb{H}^1_Q$. Then, $u \in C\big([0,T], \mathbb{H}^1_Q\big)$.*

Proof. By theorem 2.4 there is a $0 < T_0 \leqslant T$ such that $u \in C\big([0,T_0], \mathbb{H}^1_Q\big)$. Let us denote, $v := \sqrt{H+N}u$, $0 \leqslant t \leqslant T_0$. Multiplying both sides of (2.9) by $\sqrt{H+N}$ we obtain that

$$v(t) = \mathrm{e}^{-\mathrm{i}tH}(\sqrt{H+N})\varphi + \frac{1}{\mathrm{i}} G(V_1 u, u)[(H+N)^{1/2} V_2 (H+N)^{-1/2}]v. \tag{2.23}$$

Note that as $V_2 \in W_{1,\infty}$, $[(H+N)^{1/2} V_2 (H+N)^{-1/2}]$ is a bounded operator in L^2.

Equation (2.23) is a linear equation for v, where u is a fixed function in $\mathcal{N}$. Solving this equation in an interval $[T_0, T_0 + \Delta]$, with Δ small enough, we prove that $v(t) \in L^2$, for $T_0 \leqslant t \leqslant T_0 + \Delta$. Note that the length of Δ depends only on $\|u\|_{\mathcal{N}}$. Repeating this argument, step by step, we prove that $v \in C([0,T], L^2)$ and, in consequence, that $u \in C\big([0,T], \mathbb{H}^1_Q\big)$. □

In the following proposition we prove regularity between $\mathbb{H}^1_Q$ and $\mathbb{H}^2_Q$ solutions.

Proposition 2.11. *Suppose that assumption A holds that V_1 satisfies (2.7) and that $V_2 \in W_{2,\infty}$. Let $u \in C\big([0,T], \mathbb{H}^1_Q\big)$, $0 < T < \infty$, be a solution to (2.1) with $u(0) = \varphi \in \mathbb{H}^2_Q$. Then, $u \in C\big([0,T], \mathbb{H}^2_Q\big)$.*

Proof. By theorem 2.7 there is a $0 < T_0 \leqslant T$ such that $\in C\big([0,T_0], \mathbb{H}^2_Q\big)$ and then, by (2.1) $v := \frac{\partial}{\partial t} u(t) \in C_B([0,T_0], L^2)$. Moreover, taking the derivative in time of (2.9) we obtain that

$$\mathrm{i}v = \mathrm{e}^{-\mathrm{i}tH}[H\varphi + F(\varphi)] + G(2\,\{\mathrm{Re}(v, V_1 u)\} V_2 u + (V_1 u, u) V_2 v). \tag{2.24}$$

Solving the real-linear equation (2.24)—where now u is a fixed function in $\mathcal{N}^1$—in an interval $[T_0, T_0 + \Delta]$, with Δ small enough, we prove that $v(t) \in L^2$ for $T_0 \leqslant t \leqslant T_0 + \Delta$. Note that as V_1 satisfies (2.7), it follows from (2.4) that $\|V_1 u(t)\|^2_{L^2} = \big(V_1^2 u(t), u(t)\big) \leqslant C\|u\|^2_{\mathcal{N}^1}$. In consequence, the length of Δ depends only on $\|u\|_{\mathcal{N}^1}$. Repeating this argument, step by step, we prove that $v \in C([0,T], L^2)$ and, in consequence, that $u \in C\big([0,T], \mathbb{H}^2_Q\big)$. □

Let us now consider the existence of global L^2 solutions. For this purpose we prove that the L^2 norm is constant.

Lemma 2.12. *Suppose that assumption A holds that $V_1 \in L^\infty$, that $V_2 \in W_{1,\infty}$, and, furthermore, that V_1, V_2 are real valued. Then, the L^2 norm of the solution to (2.1) given by theorem 2.1 and the L^2 norm of the $\mathbb{H}^1_Q$ solution given by theorem 2.4 are constant.*

Proof. We first prove the theorem for the solution $u \in C\big([0,T], \mathbb{H}^1_Q\big)$. By (2.9) $u \in C^1\big([0,T], \mathbb{H}^{-1}_Q\big)$ and then, by (2.1),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2} = \mathrm{Re}\left(\frac{\mathrm{d}}{\mathrm{d}t}u(t), u(t)\right) = \mathrm{Re}\,\frac{1}{\mathrm{i}}[(Hu,u) + (V_1 u, u)(V_2 u, u)] = 0, \tag{2.25}$$

and then, $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$. The result in the case of solutions $u \in C([0,T], L^2)$ follows approximating $u(0) = \varphi$ in the L^2 norm by $\varphi_n \in \mathbb{H}^1_Q$ and applying theorem 2.2. □

Theorem 2.13. *Suppose that assumption A holds that $V_1 \in L^\infty$, that $V_2 \in W_{1,\infty}$, and, furthermore, that V_1, V_2 are real valued. Then, the L^2 solution, $u(t)$, given by theorem 2.1 exists for all times, and $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}, t \in [0, \infty)$.*

Proof. The theorem follows from remark 2.3 and lemma 2.12.

For $u(t) \in C\big(I, \mathbb{H}^1_Q\big)$ we define the energy at time t as follows:

$$E(u(t)) := (u'(t), u'(t)) + (V_0 u(t), u(t)) + \tfrac{1}{2}(V_1 u(t), u(t))(V_2 u(t), u(t)). \tag{2.26}$$

□

Lemma 2.14. *Suppose that assumption A holds that $V_1 = \lambda V_2$, for some real λ, that V_2 is real valued, and that $V_2 \in W_{2,\infty}$. Then, the energy of the $\mathbb{H}^1_Q$ solution to (2.1) given by theorem 2.4 and the energy of the $\mathbb{H}^2_Q$ solution to (2.1) given by theorem 2.7 are constant in time.*

Proof. We first prove the lemma for solutions $u \in C\big([0, T], \mathbb{H}^2_Q\big)$. It follows from (2.9) that $u \in C^1([0, T], L^2)$. As $\mathbb{H}^2_Q = D(H)$, we can write the energy as follows:

$$E(u(t)) := (u(t), Hu(t)) + \frac{\lambda}{2}(V_2 u(t), u(t))^2.$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} E(u(t)) &= 2\,\mathrm{Re}[(\dot{u}(t), Hu(t)) + \lambda(V_2 u(t), u(t))(V_2 \dot{u}(t), u(t))] \\ &= 2\,\mathrm{Re}\frac{1}{\mathrm{i}}[\|Hu(t)\|^2_{L^2} + \lambda(V_2 u(t), u(t))\{(V_2 u(t), Hu(t)) + (Hu(t), V_2 u(t))\}] = 0.\end{aligned}$$

The result in the case of solutions $u \in C\big([0, T], \mathbb{H}^1_Q\big)$ follows approximating $u(0) = \varphi$ in the $\mathbb{H}^1_Q$ norm by $\varphi_n \in \mathbb{H}^2_Q$ and applying theorem 2.5. □

Theorem 2.15. *Suppose that assumption A holds that $V_1 = \lambda V_2$, for some real λ, that V_2 is real valued, and that $V_2 \in W_{2,\infty}$. Then, the $\mathbb{H}^1_Q$ solution to (2.1) given by theorem 2.4 and the $\mathbb{H}^2_Q$ solution to (2.1) given by theorem 2.7 exist for all times and the L^2 norm and the energy of the solutions are constant in time.*

Proof. Let us first consider the solution $u \in C\big(I, \mathbb{H}^1_Q\big)$. By lemmata 2.12 and 2.14

$$(u'(t), u'(t)) + (V_0 u(t), u(t)) + N(u(t), u(t)) \leqslant E(u(0)) + N\|u(0)\|^2_{L^2} + \frac{|\lambda|}{2}\|V_2\|^2_{L^\infty}\|u(0)\|^4_{L^2}.$$

Then, by remark 2.6 u exists for all times, and the L^2 norm and the energy are constant in time. The theorem follows in the case of $\mathbb{H}^2_Q$ solutions by proposition 2.11. □

3. Scattering

In this section we construct the small amplitude scattering operator, S, for equation (1.2) and we give a method for the unique reconstruction of the potential V_0 and the coupling constant λ, from S.

We first introduce some standard notations and some results that we need.

For any $\gamma \in \mathbb{R}$ let us denote by L^1_γ the Banach space of all complex-valued measurable functions on $\mathbb{R}$ such that

$$\|\varphi\|_{L^1_\gamma} := \int |\varphi(x)|(1 + |x|)^\gamma \,\mathrm{d}x < \infty. \tag{3.1}$$

If $V_0 \in L^1_1$ and V_0 is real, the differential expression $\tau := -\frac{d^2}{dx^2} + V_0(x)$ is essentially self-adjoint on the domain

$$D(\tau) := \{\varphi \in L^2_C : \varphi, \text{ and } \varphi' \text{are absolutely continuous and } \tau\varphi \in L^2\},$$

where L^2_C denotes the set of all functions in L^2 that have compact support. We denote by H the unique self-adjoint realization of τ. As is well known [12, 50], H has a finite number of negative eigenvalues, it has no positive of zero eigenvalues, it has no singular-continuous spectrum, and the absolutely-continuous spectrum is $[0, \infty)$. By H_0 we denote the unique self-adjoint realization of $-\frac{d^2}{dx^2}$ with domain $\mathbb{H}^2$. The wave operators are defined as follows:

$$W_\pm := \text{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}tH_0}.$$

The limits above exist in the strong topology in L^2 and range $W_\pm = \mathcal{H}_{\mathrm{ac}}$, where $\mathcal{H}_{\mathrm{ac}}$ denotes the space of absolute continuity of H. Moreover, the intertwining relations hold $HW_\pm = W_\pm H_0$; for these results see [36]. The linear scattering operator is defined as

$$S_L := W_+^* W_-. \tag{3.2}$$

For any pair, φ, ψ, of solutions to the stationary Schrödinger equation

$$-\frac{d^2}{dx^2}\varphi + V_0\varphi = k^2\varphi, k \in \mathbb{C}, \tag{3.3}$$

let $[\varphi, \psi]$ denote the Wronskian of φ and ψ,

$$[\varphi, \psi] := \varphi'\psi - \varphi\psi'.$$

Let $f_j(x, k)$, $j = 1, 2$, be the Jost solutions to (3.3) that satisfy $f_1(x, k) \approx \mathrm{e}^{\mathrm{i}kx}, x \to \infty$, $f_2(x, k) \approx \mathrm{e}^{-\mathrm{i}kx}, x \to -\infty$ [9, 12–14]. The potential V_0 is said to be *generic* if $[f_1(x, 0), f_2(x, 0)] \neq 0$, and it is said to be *exceptional* if $[f_1(x, 0), f_2(x, 0)] = 0$. When V_0 is *exceptional* there is a bounded solution to (3.3) with $k^2 = 0$, that is called a half-bound state or a zero energy resonance. The trivial potential $V_0 = 0$ is exceptional.

Below we will always assume that $V_0 \in L^1_\gamma$, where in the *generic case* $\gamma > 3/2$ and in the *exceptional case* $\gamma > 5/2$.

In theorem 1.1 of [40] it was proven that the operators $W_\pm$ and $W_\pm^*$ are bounded on $W_{j,p}$, $j = 0, 1, 1 < p < \infty$.

By theorem 3 in page 135 of [37],

$$\|\mathcal{F}^{-1}(1 + q^2)^{j/2}(\mathcal{F}f)(q)\|_{L^p} \tag{3.4}$$

is a norm that is equivalent to the norm of $W_{j,p}$, $1 < p < \infty$. In (3.4) $\mathcal{F}$ denotes the Fourier transform.

If H has no eigenvalues the $W_\pm$ are unitary operators on L^2, and it follows from the intertwining relations that

$$(1 + H)^{j/2} = W_\pm(1 + H_0)^{j/2}W_\pm^*,$$

and then, by (3.4),

$$\|(I + H)^{j/2}f\|_{L^p}, \tag{3.5}$$

defines a norm that is equivalent to the norm of $W_{j,p}$, $j = 0, 1, 1 < p < \infty$. Below we use this equivalence without further comments. Furthermore, the following $L^p - L^{p'}$ estimate holds

$$\|\mathrm{e}^{-\mathrm{i}tH}\|_{\mathcal{B}(W_{1,p},W_{1,p'})} \leqslant C\frac{1}{|t|^{\frac{1}{p}-\frac{1}{2}}}, 1 \leqslant p \leqslant 2, 1/p + 1/p' = 1, \tag{3.6}$$

where for any pair of Banach spaces $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{B}(\mathcal{X},\mathcal{Y})$ denotes the Banach space of all bounded operators from $\mathcal{X}$ into $\mathcal{Y}$. When H has bound states estimate (3.6) is proven in [41] for the restriction of $\mathrm{e}^{-\mathrm{i}tH}$ to the subspace of continuity of H.

The norm (3.5) for the Sobolev spaces $W_{j,p}$, $1 < p < \infty$, and the $L^p - L^{p'}$ estimate (3.6) are the basic tools that we use in order to construct the scattering operator and to solve the inverse scattering problem.

For any $1 < q < 3/2$ we denote $p := (q+1)/(q-1)$, $r := (4q)/(2q-1)$. We designate $P := (1/r, 1/(p+1))$ and we define

$$L(P) := L^r(\mathbb{R}, L^{p+1}).$$

Let $\mathcal{M}$ be the following Banach space,

$$\mathcal{M} := C_B(\mathbb{R}, L^{p+1}) \cap L(P), \tag{3.7}$$

with norm

$$\|\varphi\|_{\mathcal{M}} := \max[\|\varphi\|_{C_B(\mathbb{R},L^{p+1})}, \|\varphi\|_{L(P)}].$$

Recall that $C_B(\mathbb{R}, L^{p+1})$ denotes the Banach space of all bounded and continuous functions from $\mathbb{R}$ into L^{p+1}.

In the following theorem we construct the small amplitude nonlinear scattering operator.

Theorem 3.1. *Suppose that $V_0 \in L^1_\gamma$ where in the generic case $\gamma > 3/2$ and in the exceptional case $\gamma > 5/2$ and that H has no eigenvalues. Moreover, assume that $V_j \in L^q \cap L^\infty$ for some $1 < q < 3/2$. Then, there is a $\delta > 0$ such that for every $\varphi_- \in \mathbb{H}^1 \cap L^{1+\frac{1}{p}}$ with $\|\varphi_-\|_{\mathbb{H}^1} + \|\varphi_-\|_{L^{1+\frac{1}{p}}} < \delta$ there is a unique solution, u, to (1.2) such that $u \in \mathcal{M} \cap C_B(\mathbb{R}, L^2)$ and*

$$\lim_{t\to-\infty} \|u(t) - \mathrm{e}^{-\mathrm{i}tH}\varphi_-\|_{L^2} = 0. \tag{3.8}$$

Moreover, there is a unique $\varphi_+ \in L^2$ such that

$$\lim_{t\to\infty} \|u(t) - \mathrm{e}^{-\mathrm{i}tH}\varphi_+\|_{L^2} = 0. \tag{3.9}$$

Furthermore, $\mathrm{e}^{-\mathrm{i}tH}\varphi_\pm \in \mathcal{M}$ and

$$\|u(t) - \mathrm{e}^{-\mathrm{i}tH}\varphi_\pm\|_{\mathcal{M}} \leqslant C\|\mathrm{e}^{-\mathrm{i}tH}\varphi_\pm\|^3_{\mathcal{M}}, \tag{3.10}$$

$$\|\varphi_+ - \varphi_-\|_{L^2} \leqslant C\big[\|\varphi_-\|_{\mathbb{H}^1} + \|\varphi_-\|^3_{L^{1+\frac{1}{p}}}\big]. \tag{3.11}$$

The scattering operator $S_{V_0} : \varphi_- \hookrightarrow \varphi_+$ is injective.

Proof. Observe that $u \in C_B(\mathbb{R}, L^2) \cap \mathcal{M}$ is a solution to (1.2) with $\lim_{t\to-\infty} \|u(t) - \mathrm{e}^{-\mathrm{i}tH_0}\varphi_-\|_{L^2} = 0$ if and only if it is a solution to the following integral equation (this is proven as in the proof of the equivalence of (2.1) and (2.9))

$$u = \mathrm{e}^{-\mathrm{i}tH}\varphi_- + \frac{1}{\mathrm{i}}\int_{-\infty}^{t} \mathrm{e}^{-\mathrm{i}(t-\tau)H} F(u(\tau))\,\mathrm{d}\tau, \tag{3.12}$$

where

$$F(u) := \lambda(V_1u, u)V_2u. \tag{3.13}$$

We will prove that the integral in the right-hand side of (3.12) is absolutely convergent in $\mathcal{M}$ and in L^2. □

For $u \in \mathcal{M}$ we define

$$\mathcal{P}(u)(t) := \int_{-\infty}^{t} \mathrm{e}^{-\mathrm{i}(t-\tau)H} F(u(\tau))\, \mathrm{d}\tau. \tag{3.14}$$

By (3.6) and Hölder's inequality,

$$\begin{aligned} \|(\mathcal{P}u)(t)\|_{L^{p+1}} &\leqslant C \int_{-\infty}^{t} \frac{1}{|t-\tau|^d} \|F(u(\tau))\|_{L^{\frac{p}{p+1}}} \\ &\leqslant C \int_{-\infty}^{t} \frac{1}{|t-\tau|^d} \|u(\tau)\|^3_{L^{p+1}}\, \mathrm{d}\tau, \end{aligned} \tag{3.15}$$

where $d := 1/(2q)$. Then,

$$\|(\mathcal{P}u)(t)\|_{L^{p+1}} \leqslant C(I_1 + I_2), \tag{3.16}$$

where

$$I_1 := \int_{-\infty}^{t-1} \frac{1}{|t-\tau|^d} \|u(\tau)\|^3_{L^{p+1}}\, \mathrm{d}\tau,$$

and

$$I_2 := \int_{t-1}^{t} \frac{1}{|t-\tau|^d} \|u(\tau)\|^3_{L^{p+1}}\, \mathrm{d}\tau.$$

Let us denote by $\chi_{(1,\infty)}$ the characteristic function of $(1,\infty)$. Then, by Hölder's inequality,

$$I_1 \leqslant \left\| \chi_{(1,\infty)}(\tau) \frac{1}{|\tau|^d} \right\|_{L^\alpha} \|u\|^3_{L(P)},$$

where $\alpha := r/(r-3)$. Note that as $\mathrm{d}\alpha > 1$, $\|\chi_{(1,\infty)}(\tau)\frac{1}{|\tau|^d}\|_{L^\alpha} < \infty$.

Moreover, as $d < 1$,

$$I_2 \leqslant C\|u\|^3_{C_B(\mathbb{R},L^{p+1})}.$$

It follows that

$$\|(\mathcal{P}u)(t)\|_{L^{p+1}} \leqslant C\|u\|^3_{\mathcal{M}}. \tag{3.17}$$

We prove in a similar way that $(\mathcal{P}u)(\cdot)$ is a continuous function on $\mathbb{R}$ with values in L^{p+1}.

Furthermore, it follows from (3.15) and the generalized Young inequality [31] that

$$\|\mathcal{P}(u)\|_{L(P)} \leqslant C\|u\|^3_{L(P)}. \tag{3.18}$$

Then, by (3.17) and (3.18),

$$\|\mathcal{P}u\|_{\mathcal{M}} \leqslant C\|u\|^3_{\mathcal{M}}. \tag{3.19}$$

In an analogous way we prove that

$$\|\mathcal{P}u - \mathcal{P}v\|_{\mathcal{M}} \leqslant C\left(\|u\|^2_{\mathcal{M}} + \|v\|^2_{\mathcal{M}}\right)\|u-v\|_{\mathcal{M}}. \tag{3.20}$$

Furthermore,

$$\begin{aligned} \|(\mathcal{P}u)(t)\|^2_{L^2} &= \int_{-\infty}^{t} \mathrm{d}\tau \left(F(u(\tau)), \int_{-\infty}^{t} \mathrm{e}^{-\mathrm{i}(\tau-\tau')H} F(u(\tau')) \right) \\ &= 2\,\mathrm{Re} \int_{-\infty}^{t} (F(u(\tau)), \mathcal{P}(u)(\tau))\, \mathrm{d}\tau. \end{aligned}$$

We define

$$u_t(\tau) := \chi_{(-\infty,t)}(\tau)u(\tau), \qquad g_t(\tau) := \int_{-\infty}^{\tau} \frac{1}{|\tau-\tau'|^d} \|u_t(\tau')\|^3_{L^{p+1}}\, \mathrm{d}\tau.$$

Then, by (3.6) and Hölder's inequality,

$$\|(\mathcal{P}u)(t)\|_{L^2}^2 \leqslant \int_{-\infty}^{t} \|u_t(\tau)\|_{L^{p+1}}^3 g_t(\tau)\,\mathrm{d}\tau, \leqslant C\left[\int \|u_t(\tau)\|_{L^{p+1}}^{3r/(r-1)}\,\mathrm{d}\tau\right]^{(r-1)/r} \|g_t\|_{L^r}. \quad (3.21)$$

By the generalized Young's inequality,

$$\|g_t\|_{L^r} \leqslant C\|u_t\|_{L(P)}^3.$$

Furthermore,

$$\left[\int \|u_t(\tau)\|_{L^{p+1}}^{3r/(r-1)}\,\mathrm{d}\tau\right]^{(r-1)/r} \leqslant \|u\|_{\mathcal{M}}^{4-r}\|u_t\|_{L(P)}^{r-1}.$$

Hence, by (3.21)

$$\|(\mathcal{P}u)(t)\|_{L^2}^2 \leqslant \|u\|_{\mathcal{M}}^{4-r}\|u_t\|_{\mathcal{M}}^{r+2}. \quad (3.22)$$

We prove in a similar way that the function $t \in \mathbb{R} \to (\mathcal{P}u)(t)$ with values in L^2 is continuous.

We first prove that equation (3.12) has at most one solution in $\mathcal{M}$ and then, we prove the existence of a solution for φ small.

Suppose that there are two solutions in $\mathcal{M}$ to (3.12), u, v, and denote, $u_T := \chi_{(-\infty,T)}u$, $v_T := \chi_{(-\infty,T)}v$. Then,

$$u_T(t) - v_T(t) = (\mathcal{P}u_T)(t) - (\mathcal{P}v_T)(t), \text{ for } t \leqslant T. \quad (3.23)$$

Arguing as in the proof of (3.18), and as $u_T(t) = v_T(t) = 0$ for $t \geqslant T$, we prove that

$$\|u_T - v_T\|_{L(P)} \leqslant C\left(\|u_T\|_{L(P)}^2 + \|v_T\|_{L(P)}^2\right)\|u_T - v_T\|_{L(P)}. \quad (3.24)$$

As $\lim_{T\to-\infty}\left(\|u_T\|_{L(P)}^2 + \|v\|_{L(P)}^2\right) = 0$ we can take T so negative that

$$C\left(\|u_T\|_{L(P)}^2 + \|v\|_{L(P)}^2\right) < 1/2,$$

where C is the constant in (3.24). Then, for such T equation (3.24) implies that

$$\|u_T - v_T\|_{L(P)} < \frac{1}{2}\|u_T - v_T\|_{L(P)},$$

and then, $u_T(t) = v_T(t)$, $t \leqslant T$, and by the uniqueness of the initial value problem at finite time (see theorem 2.1), $u(t) = v(t)$, $t \in \mathbb{R}$.

We now prove that $\mathrm{e}^{-\mathrm{i}tH} \in \mathcal{B}(\mathbb{H}^1 \cap L^{1+\frac{1}{p}}, \mathcal{M})$.

Since $\mathrm{e}^{-\mathrm{i}tH}$ is a strongly continuous unitary group on L^2 that commutes with $(1+H)^{1/2}$ we have that $\mathrm{e}^{-\mathrm{i}tH} \in \mathcal{B}(\mathbb{H}^1, C_B(\mathbb{R}, \mathbb{H}^1))$. Furthermore, as by Sobolev's theorem [5] and interpolation [31] $\mathbb{H}^1$ is continuously imbedded in L^{p+1}, it follows that $\mathrm{e}^{-\mathrm{i}tH} \in \mathcal{B}(\mathbb{H}^1, C_B(\mathbb{R}, L^{p+1}))$.

Moreover, by the $L^p - L^{p'}$ estimate (3.6), $\mathrm{e}^{-\mathrm{i}tH} \in \mathcal{B}(L^{1+\frac{1}{p}}, L^r(\mathbb{R}\backslash[-1,1], L^{p+1}))$. Combining these two results we have that

$$\mathrm{e}^{-\mathrm{i}tH} \in \mathcal{B}(\mathbb{H}^1 \cap L^{1+\frac{1}{p}}, \mathcal{M}). \quad (3.25)$$

As in section 2, for $R > 0$ we denote

$$\mathcal{M}_R := \{u \in \mathcal{M} : \|u\|_{\mathcal{M}} \leqslant R\}.$$

We now take R so small that $C\max[R^3, (2R)^2] < 1/2$, where C is the biggest of the constants in (3.19) and (3.20), and δ so small that

$$\|\mathrm{e}^{-\mathrm{i}tH}\varphi\|_{\mathcal{M}} \leqslant R/4, \qquad \text{for} \quad \|\varphi\|_{\mathbb{H}^1} + \|\varphi\|_{L^{1+\frac{1}{p}}} < \delta.$$

Then if $\|\varphi\|_{\mathbb{H}^1} + \|\varphi\|_{L^{1+\frac{1}{p}}} < \delta$ the operator

$$\mathcal{C}(u) := \mathrm{e}^{-\mathrm{i}tH}\varphi + \mathcal{P}(u) \quad (3.26)$$

is a contraction on $\mathcal{M}_R$. By the contraction mapping theorem [30] $\mathcal{C}$ has a unique fixed point in $\mathcal{M}_R$ that is a solution to (3.12), and moreover,

$$\|u\|_{\mathcal{M}} \leqslant \|\,\mathrm{e}^{-\mathrm{i}tH}\varphi\|_{\mathcal{M}} + \tfrac{1}{2}\|u\|_{\mathcal{M}}, \tag{3.27}$$

and hence,

$$\|u\|_{\mathcal{M}} \leqslant 2\|\,\mathrm{e}^{-\mathrm{i}tH}\varphi_-\|_{\mathcal{M}}. \tag{3.28}$$

By (3.12) and (3.22), $u \in C_B(\mathbb{R}, L^2)$ and (3.8) holds. Equations (3.12), (3.19) and (3.28) imply that (3.10) holds for φ_-. We define

$$\varphi_+ := \varphi_- + \frac{1}{\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\tau H} F(u(\tau))\,\mathrm{d}\tau. \tag{3.29}$$

Estimating as in the proof of (3.17) we prove that $\varphi_+ \in L^{p+1}$, and arguing as in the proof of (3.22) it follows that $\varphi_+ \in L^2$ and that

$$\|\varphi_+ - \varphi_-\|_{L^2} \leqslant C\|u\|_{\mathcal{M}}^3 \leqslant C\|\mathrm{e}^{-\mathrm{i}tH}\varphi\|_{\mathcal{M}}^3.$$

Equation (3.11) follows now from (3.25).

By (3.12) and (3.29),

$$u = \mathrm{e}^{-\mathrm{i}tH}\varphi_+ - \frac{1}{i}\int_t^{\infty} \mathrm{e}^{-\mathrm{i}(t-\tau)H} F(u(\tau))\,\mathrm{d}\tau. \tag{3.30}$$

Equation (3.9) follows from (3.30) estimating as in the proof of (3.22). Moreover, estimating as in the proof of (3.19) we have that

$$\left\| \int_t^{\infty} \mathrm{e}^{-\mathrm{i}(t-\tau)H} F(u(\tau))\,\mathrm{d}\tau \right\|_{\mathcal{M}} \leqslant C\|u\|_{\mathcal{M}}^3. \tag{3.31}$$

Multiplying both sides of (3.29) by $\mathrm{e}^{-\mathrm{i}tH}$ and estimating as in the proof of (3.19) we prove that $\mathrm{e}^{-\mathrm{i}tH}\varphi_+ \in \mathcal{M}$. By (3.30), (3.31) and arguing as in the proof of (3.28) we obtain that

$$\|u\|_{\mathcal{M}} \leqslant 2\|\,\mathrm{e}^{-\mathrm{i}tH}\varphi_+\|_{\mathcal{M}}. \tag{3.32}$$

At this point, (3.10) for φ_+ follows from (3.30)–(3.32). Note that the uniqueness of φ_+ is immediate from the fact that $\mathrm{e}^{-\mathrm{i}tH}$ is unitary on L^2.

Finally, we prove that S_{V_0} is injective. Suppose that $\varphi_+ = S_{V_0}\varphi_- = 0$. Then, by (3.30)

$$u = -\frac{1}{\mathrm{i}}\int_t^{\infty} \mathrm{e}^{-\mathrm{i}(t-\tau)H} F(u(\tau))\,\mathrm{d}\tau. \tag{3.33}$$

We prove that (3.33) implies that $u = 0$ arguing as in the proof of the uniqueness of the solution to equation (3.12) and then, it follows from (3.8) that $\varphi_- = 0$.

We now define the scattering operator that relates asymptotic states that are solutions to the free Schrödinger equation,

$$\mathrm{i}\frac{\partial}{\partial t}u = H_0 u,$$

given by

$$S := W_+^* S_{V_0} W_-. \tag{3.34}$$

In the following theorem we show that we can uniquely reconstruct the linear scattering operator from the small amplitude behaviour of S.

Theorem 3.2. *Suppose that the assumptions of theorem 3.1 are satisfied. Then, for every* $\varphi \in \mathbb{H}^1 \cap L^{1+\frac{1}{p}}$,

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} S(\epsilon\varphi)|_{\epsilon=0} = S_L\varphi, \tag{3.35}$$

where the derivative exists in the strong convergence in L^2.

Proof. Since $S(0) = 0$ and the wave operators $W_\pm$ are bounded on $\mathbb{H}^1$ [40] it is sufficient to prove that

$$\text{s}-\lim_{\epsilon\to 0}\frac{1}{\epsilon}[S_{V_0}(\epsilon\varphi)-\epsilon\varphi]=0. \tag{3.36}$$

But (3.36) follows from (3.11) with φ_- replaced by $\epsilon\varphi$. □

Corollary 3.3. *Suppose that the assumptions of theorem 3.1 are satisfied. Then, S uniquely determines the linear potential* V_0.

Proof. By theorem 3.2 we uniquely reconstruct S_L from S. From S_L we obtain the reflection coefficients for linear Schrödinger scattering on the line (see section 9.7 of [26]). As H has no bound states we uniquely reconstruct V_0 from one of the reflection coefficients using any of the standard methods. See, for example [6, 9, 12–14, 24, 25]. □

Note that our proof gives a constructive method to uniquely reconstruct V_0 from S. We first compute S_L from the derivative in (3.35) and then, we obtain the reflection coefficients and we reconstruct V_0 from one of them.

The following theorem gives us a convergent expansion at low amplitude for S_{V_0}.

Theorem 3.4. *Suppose that the assumptions of theorem 3.1 are satisfied. Then, for any* $\varphi\in\mathbb{H}^1\cap L^{1+\frac{1}{p}}$,

$$\mathrm{i}((S_{V_0}-I)(\epsilon\varphi),\varphi)=\epsilon^3\lambda\int_{-\infty}^{\infty}(V_1\,\mathrm{e}^{-\mathrm{i}tH}\varphi,\mathrm{e}^{-\mathrm{i}tH}\varphi)(V_2\,\mathrm{e}^{-\mathrm{i}tH}\varphi,\mathrm{e}^{-\mathrm{i}tH}\varphi)\,\mathrm{d}t+O(\epsilon^5),\quad \epsilon\to 0. \tag{3.37}$$

Proof. Suppose that $u_j\in\mathcal{M}$, $j=1,\ldots,4$. Then, by Hölder's inequality,

$$\begin{aligned}
&\left|\int(V_1u_1(t),u_2(t))(V_2u_3(t),u_4(t))\,\mathrm{d}t\right|\\
&\quad\leqslant\|V_1\|_{L^q}\|V_2\|_{L^q}\int\|u_1(t)\|_{L^{p+1}}\|u_2(t)\|_{L^{p+1}}\|u_3(t)\|_{L^{p+1}}\|u_4(t)\|_{L^{p+1}}\,\mathrm{d}t\\
&\quad\leqslant\|V_1\|_{L^q}\|V_2\|_{L^q}\|u_1\|_{L(P)}\|u_2\|_{L(P)}\|u_3\|_{L(P)}\left[\int\|u_4(t)\|_{L^{p+1}}^{r/(r-3)}\,\mathrm{d}t\right]^{(r-3)/r}\\
&\quad\leqslant\|V_1\|_{L^q}\|V_2\|_{L^q}\|u_1\|_{L(P)}\|u_2\|_{L(P)}\|u_3\|_{L(P)}\|u_4\|_{\mathcal{M}}^{4-r}\|u_4\|_{L(P)}^{r-3}\\
&\quad\leqslant\|V_1\|_{L^q}\|V_2\|_{L^q}\Pi_{j=1}^4\|u_j\|_{\mathcal{M}}.
\end{aligned} \tag{3.38}$$

As $\mathrm{e}^{-\mathrm{i}tH}\varphi\in\mathcal{M}$, (3.38) with $u_j=\mathrm{e}^{-\mathrm{i}tH}\varphi$, $j=1,2,3,4$, proves that the integral in the right-hand side of (3.37) is absolutely convergent.

By the contraction mapping theorem, the solution u that satisfies (3.8) with $\epsilon\varphi$ instead of φ_- is given by

$$u(t)=\lim_{j\to\infty}\mathcal{C}^j(\epsilon\,\mathrm{e}^{-\mathrm{i}tH}\varphi)=\epsilon\,\mathrm{e}^{-\mathrm{i}tH}\varphi+v(t),\text{ where } v(t)=\sum_{j=1}^{\infty}\mathcal{P}^j(\epsilon\,\mathrm{e}^{-\mathrm{i}tH}\varphi). \tag{3.39}$$

Moreover, it follows from (3.19) that if ϵ is small enough,

$$\|v\|_{\mathcal{M}}\leqslant C\epsilon^3. \tag{3.40}$$

Then, (3.37) follows by (3.29) with $\epsilon\varphi$ instead of φ_- and (3.38)–(3.40). □

Corollary 3.5. *Suppose that the conditions of theorem 3.1 are satisfied and that V_1, V_2 are real-valued functions that are not identically zero. Moreover, assume either that $V_1 = V_2$ or that V_j, $j = 1, 2$, do not change sign and $V_1 V_2 \neq 0$ in a set of positive measure. Then, the scattering operator, S, and V_j, $j = 1, 2$, determine uniquely λ.*

Proof. By (3.37)

$$\lambda = \lim_{\epsilon\to 0} \frac{1}{\epsilon^3} \frac{\mathrm{i}((S_{V_0} - I)(\epsilon\varphi), \varphi)}{\int_{-\infty}^{\infty} (V_1\, \mathrm{e}^{-\mathrm{i}tH}\varphi, \mathrm{e}^{-\mathrm{i}tH}\varphi)(V_2\, \mathrm{e}^{-\mathrm{i}tH}\varphi, \mathrm{e}^{-\mathrm{i}tH}\varphi)\,\mathrm{d}t}. \tag{3.41}$$

By corollary 3.3 V_0 is known, and then $H := H_0 + V_0$ is known. Then, $W_\pm$ are known, and S uniquely determines S_{V_0} (see (3.34)). Hence, the right-hand side of (3.41) is uniquely determined by our data. Moreover, under our conditions we can always find a $\varphi \in \mathbb{H}^1$ such that the denominator of the right-hand side of(3.41) is not zero. □

Note that (3.41) gives us a formula for the reconstruction of λ.

Let us now go back to the quantum capacitor (1.1) where we take a slightly more general external potential V_0, namely,

$$V_0(x) = [\beta_1 \chi_{[a,b]}(x) + \beta_2 \chi_{[c,d]}(x)],$$

where $\beta_1, \beta_2 \in \mathbb{R}$. By corollary 3.3 we uniquely reconstruct V_0 from S. Then, $\beta_1, \beta_2, a, b, c$ and d are uniquely reconstructed. Moreover, by corollary 3.5 we uniquely reconstruct λ. Hence, from S we uniquely reconstruct all the physical parameters of the quantum capacitor.

References

[1] Adami R and Teta A 2001 A class of nonlinear Schrödinger equations with concentrated nonlinearities *J. Funct. Anal.* **180** 148–75

[2] Adami R 2002 Blow-up for Schrödinger equations with pointwise nonlinearity *Mathematical Methods in Quantum Mechanics (Contemporary Mathematics* vol 307*)* ed R Weder, P Exner and B Grébert (Providence, RI: American Mathematical Society) pp 1–7

[3] Adami R, Dell'Antonio G, Figari R and Teta A 2003 The Cauchy problem for the Schrödinger equation in dimension three with concentrated nonlinearity *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20** 477–500

[4] Adami R, Dell'Antonio G, Figari R and Teta A 2004 Blow up solutions for the Schrödinger equation in dimension three with concentrated nonlinearity *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** 121–37

[5] Adams R A 1975 *Sobolev Spaces* (New York: Academic)

[6] Aktosun T and Klaus M 2002 Inverse theory: problem on the line *Scattering* vol 1 ed R Pike and P Sabatier (New York: Academic) pp 770–85

[7] Altshuler B L, Lee P A and Webb R A 1991 *Mesoscopic Phenomena in Solids* (New York: Elsevier)

[8] Bourgain J 1999 *Global Solutions of Nonlinear Schrödinger Equations (Colloquiun Publications* vol 46*)* (Providence, RI: American Mathematical Society)

[9] Chadam K and Sabatier P C 1989 *Inverse Problems in Quantum Scattering Theory* 2nd edn (Berlin: Springer)

[10] Davies E B 1979 Symmetry breaking for a nonlinear Schrödinger operator *Commun. Math. Phys.* **64** 191–210

[11] Davies E B 1995 Nonlinear Schrödinger operator and molecular structure *J. Phys. A: Math. Gen.* **28** 4025–41

[12] Deift P and Trubowitz E 1979 Inverse scattering on the line *Commun. Pure Appl. Math.* **32** 121–251

[13] Faddeev L D 1964 Properties of the S matrix of the one–dimensional Schrödinger equation *Trudy Math. Inst. Steklov* **73** 314–33
Faddeev L D 1964 *Am. Math. Soc. Transl. Ser. 2* **65** 139–66 (Engl. Transl.)

[14] Faddeev L D 1974 Inverse problems of quantum scattering theory, II *Itogi Nauki i Tekhniki Sovremennye Problemy Matematiki* **3** 93–180
Faddeev L D 1976 *J. Sov. Math.* **5** 334–96 (Engl. Transl.)

[15] Ginibre J 1998 *Introduction aux Équations de Schrödinger non Linéaires* (Paris: Onze Éditions)

[16] Grecchi V and Martinez A 1995 Non linear Stark effect and molecular localization *Commun. Math. Phys.* **166** 533–48

[17] Grecchi V, Martinez A and Sachetti A 2002 Destruction of the beating effect for a nonlinear Schrödinger equation *Commun. Math. Phys.* **227** 191–209

[18] Jona-Lasinio G, Presilla C and Capasso F 1992 Chaotic quantum phenomena without classical counterpart *Phys. Rev. Lett.* **68** 2269–72

[19] Jona-Lasinio G 1995 Stationary solutions and invariant tori for a class of nonlinear non-local Schrödinger equations *Advances in Dynamical Systems and Quantum Physics* (River Edge, NJ: World Scientific) pp 142–6

[20] Jona-Lasinio G, Presilla C and Sjöstrand J 1995 On Schrödinger equations with concentrated nonlinearities *Ann. Phys.* **240** 1–21

[21] Kato T 1976 *Perturbation Theory of Linear Operators* 2nd edn (Berlin: Springer)

[22] Kato T 1989 Nonlinear Schrödinger equations *Schrödinger Operators (Lecture Notes in Physics* vol 345*)* ed H Holden and A Jensen (Berlin: Springer) pp 218–63

[23] Kato T 1984 Remarks on holomorphic families of Schrödinger and Dirac operators *Differential Equations* ed I W Knowles and R T Lewis (Amsterdam: Elsevier) pp 341–52

[24] Marchenko V A 1986 *Sturm-Liouville Operators and Applications* (Basel: Birkhäuser)

[25] Melin A 1985 Operator methods for inverse scattering on the real line *Commun. Partial Diff. Eqns* **10** 677–766

[26] Pearson D B 1988 *Quantum Mechanics and Spectral Theory* (New York: Academic)

[27] Presilla C, Jona-Lasinio G and Capasso F 1991 Non-linear feedback oscillations in resonant tunneling through double barriers *Phys. Rev.* B **43** 5200–3

[28] Presilla C and Sjöstrand J 1996 Transport properties in resonant tunneling *J. Math. Phys.* **37** 4816–44

[29] Racke R 1992 *Lectures in Nonlinear Evolution Equations. Initial Value Problems (Aspects of Mathematics* vol E19*)* (Braunschweig: Vieweg)

[30] Reed M and Simon B 1972 *Methods of Modern Mathematical Physics I Functional Analysis* (New York: Academic)

[31] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics II Fourier Analysis, Self-Adjointness* (New York: Academic)

[32] Sachetti A 2002 Tunneling destruction for a nonlinear Schrödinger equation *Mathematical Results in Quantum Mechanics (Contemporary Mathematics* vol 307*)* ed R Weder, P Exner and B Grébert (Providence, RI: American Mathematical Society) pp 275–9

[33] Sachetti A 2004 Nonlinear Time-dependent Schrödinger equation with double well potential *Multiscale Methods in Quantum Mechanics* ed P Blanchard and G Dell'Antonio (Basel: Birkhäuser)

[34] Sachetti A 2005 Nonlinear double well Schrödinger equation in the semiclassical limit *J. Stat. Phys.* **119** 1347–81

[35] Sachetti A 2004 Nonlinear time-dependent one-dimensional Schrödinger equation with a double well potential *SIAM J. Math. Anal.* **35** 1160–76

[36] Schechter M 1981 *Operator Methods in Quantum Mechanics* (New York: North-Holland)

[37] Stein E M 1970 *Singular Integrals and Differentiability Properties of Functions* (Princeton, NJ: Princeton University Press)

[38] Strauss W 1989 *Nonlinear Wave Equations (CBMS-RCMS* vol 73*)* (Providence, RI: American Mathematical Society)

[39] Sulem C and Sulem P-L 1999 *The Nonlinear Schrödinger Equation (Appl. Math. Sciences* vol 139*)* (New York: Springer)

[40] Weder R 1999 The $W_{k,p}$ continuity of the Schrödinger wave operators on the line *Commun. Math. Phys.* **208** 507–20

[41] Weder R 2000 $L^p - L^{p'}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential *J. Funct. Anal.* **170** 37–68

[42] Weder R 2000 Inverse scattering on the line for the nonlinear Klein–Gordon equation. Reconstruction of the potential and the nonlinearity *J. Math. Anal. Appl.* **252** 102–23

[43] Weder R 2000 Center manifolds for nonintegrable nonlinear Schrödinger equations on the line *Commun. Math. Phys.* **215** 343–56

[44] Weder R 2001 Inverse scattering for the nonlinear Schrödinger equation. Reconstruction of the potential and the nonlinearity *Math. Methods Appl. Sci.* **24** 245–54

[45] Weder R 2001 Inverse scattering for the nonlinear Schrödinger equation II. Reconstruction of the potential and the nonlinearity in the multidimensional case *Proc. Am. Math. Soc.* **129** 3637–45

[46] Weder R 2002 Multidimensional inverse scattering for the nonlinear Klein–Gordon equation with a potential *J. Diff. Eqns* **184** 62–77

[47] Weder R 2003 The time dependent approach to inverse scattering *Advances in Differential Equations and Mathematical Physics (Contemporary Mathematics* vol 327*)* ed J Karpeshina, G Stolz, R Weikard and Y Zeng (Providence, RI: American Mathematical Society) pp 359–77

[48] Weder R 2005 Scattering for the forced non-linear Schrödinger equation with a potential on the half-line *Math. Methods Appl. Sci.* **28** 1219–36

[49] Weder R 2005 The forced non-linear Schrödinger equation with a potential on the half-line *Math. Methods Appl. Sci.* **28** 1237–55

[50] Weidmann J 1987 *Spectral Theory of Ordinary Differential Operators (Lecture Notes in Math. vol 1258)* (Berlin: Springer)